Convergence of Sequence

Def. A Sequence $\{S_n\}$ is said to converge to a real number l (or to have the real number l as limit) if for each $\epsilon > 0$, there exists a positive integer m (depending on ϵ) such that $|S_n - l| < \epsilon$ for all $n \geq m$

$$S_n \to l \text{ as } n \to \infty \text{ or } \lim_{n\to\infty} S_n = l$$

Some theorem

Th. I Every convergent sequence is bdd.

Let a Sequence $\{S_n\}$ converge to the limit l.
Let $\epsilon > 0$ be a Given number, so that $\exists$ a positive integer m such that

$$|S_n - l| < \epsilon \quad \forall n \geq m$$

$$\Leftrightarrow l - \epsilon < S_n < l + \epsilon \quad \forall n \geq m$$

$$g = \min\{l-\epsilon, s_1, s_2 \text{ --- } s_{m-1}\}$$

$$G = \max\{l+\epsilon, S_1, S_2 \text{ --- } S_{m-1}\}$$

Thus we have

$g \leq S_n \leq G \ \forall n$

Hence $\{S_n\}$ is a bdd Sequence.

Th 2 A Sequence can't Converge to more than one limit.

Let, if possible, a Sequence $\{S_n\}$ Converge to two real number l and l' ($l \neq l'$) Let us select

$\epsilon = \frac{1}{3}|l - l'| > 0$

Since the Sequence $\{S_n\}$ converges to l and l'; therefore, there exist positive integers m_1, m_2 such that

$|S_n - l| < \epsilon \ \forall n \geq m_1$ — (1)

and

$|S_n - l'| < \epsilon, \forall \ n \geq m_2$ (2)

Now from (1) and (2), for ~~$n \geq \max(m_1$~~

$n \geq \max(m_1, m_2)$

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$$|l-l'| = |l-S_n+S_n-l'|$$
$$\leq |l-S_n|+|S_n-l'| < 2\epsilon$$

i.e. $|l-l'| < \frac{2}{3}|l-l'|$ which is not possible.

Hence the sequence can't converge to two limits.

Cauchy's General Principle of Convergence.

Th A necessary and sufficient condition for the convergence of a sequence $\{S_n\}$ is that for each $\epsilon > 0$ there exists a positive integer m such that

$$|S_{n+p} - S_n| < \epsilon, \forall\, n \geq m \wedge p \geq 1$$

Necessary. Let the sequence be convergent and let l be limit, so that for a given $\epsilon > 0$, $\exists$ a positive integer m, such that

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$$|S_n - l| < \frac{1}{2}\epsilon \ \forall n \geq m$$

If $p \geq 1$, then $n+p > n \geq m$ and so

$$|S_{n+p} - l| < \frac{1}{2}\epsilon \ \forall n \geq m \wedge p \geq 1$$

$$\Rightarrow |S_{n+p} - S_n| = |S_{n+p} - l + l - S_n|$$

$$\cancel{\leq |S_{n+p} - S_n|}$$

$$\leq |S_{n+p} - l| + |l - S_n|$$

$$< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon \ \forall n \geq m \wedge p \geq 1$$

Sufficient To establish the convergence of the sequence as a consequence of the given conditions, we show first that the sequence is bdd and then that it converges to a limit.

Now by the given condition for $\epsilon = 1$, $\exists$ a positive integer m such that

$$|S_{n+p} - S_n| < 1, \ \forall n \geq m_0 \wedge p \geq 1$$

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In particular, for $n = m_0$

$$|S_{m_0+p} - S_{m_0}| < 1 \;\forall\, p \geq 1$$

i.e. $S_{m_0} - 1 < S_{m_0+p} < S_{m_0} + 1 \;\forall\, p \geq 1$

Let $g = \min\{S_1, S_2 \ldots S_{m_0-1}, S_{m_0} - 1\}$

$G = \max\{S_1, S_2, \ldots S_{m_0-1}, S_{m_0}+1\}$

Then $g \leq S_n \leq G \;\forall\, n$

Hence the sequence is bounded and therefore by Bolzano-Weierstrass theorem for sequences, it has at least one limit point, say l. We shall now show that the sequence converges to l, i.e. $\lim S_n = l$.

Now by the given condition, for $\epsilon > 0$ $\exists$ a positive integer m such that

$$|S_{n+p} - S_n| < \frac{1}{3}\epsilon, \;\forall\, n \geq m \wedge p \geq 1$$

In particular, for $n = m$

$$|S_{n+p} - S_m| < \frac{1}{3}\epsilon, \;\forall\, p \geq 1 \quad \text{--- (1)}$$

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As l is a limit point, $\exists$ an integer $m_1 > m$ Such that

$$|S_{m_1} - l| < \frac{1}{3}\epsilon \quad \text{--- (2)}$$

Also, since $m_1 > m$ therefore from (1) we have

$$|S_{m_1} - S_m| < \frac{1}{3}\epsilon \quad \text{--- (3)}$$

$$|S_{m+p} - l| = |S_{m+p} - S_m + S_m - S_{m_1} + S_{m_1} - l|$$

$$\leq |S_{m+p} - S_m| + |S_m - S_{m_1}| + |S_{m_1} - l|$$

$$< \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon \quad \forall p \geq 1$$

$\Rightarrow |S_n - l| < \epsilon, \forall n \geq m$

Hence the sequence $\{S_n\}$ converges to l.

Cauchy Sequence

A Sequence $\{S_n\}$ is called a Cauchy sequence or a fundamental sequence if for each $\epsilon > 0$, $\exists$ there exist a positive integer m, Such that

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$$|S_{n+p} - S_n| < \epsilon, \forall\, n \geq m \;\&\; p \geq 1$$

$$|S_{n_1} - S_{n_2}| < \epsilon \quad \forall\, n_1, n_2 \geq m$$

Thus in the field of real number a sequence is convergent iff it is a cauchy sequence.